Journal of Engineering Mathematics **27**: 89–98, 1993.

The linear steady thermoelastic problem for a strip with a collinear array of Griffith cracks parallel to its edges

STUART DAVIDSON

Department of Mathematics and Computer Science, Georgia Southern University, Statesboro, Georgia 30460-8093, USA

Received 27 November 1991; accepted in revised form 28 August 1992

Abstract. The problem of determining the thermal stresses when a uniform heat flow in a thermoelastic strip is disturbed by a collinear array of cracks is discussed. The solution of the Duhamel–Neumann equations is posed in terms of harmonic functions, which leads to dual series relations whose solutions are known. Numerical results for the stress intensity factors at the crack tips are displayed in graphical form.

1. Introduction

The problem considered here is that of determining the stress intensity factors when a collinear array of Griffith cracks disturb a uniform heat flow in an infinite, two-dimensional, linear, isotropic, thermoelastic strip whose edges are traction free and held at different temperatures. The strip which occupies the region $-\infty < X < \infty$, $-H < Y < H$ is assumed to deform under plane strain conditions while the cracks, which are thermally insulated and traction free, are assumed to form a periodic array defined by $0 < |X - 2NP| < B < P$, $N = 0, \pm 1, \pm 2, \ldots$. The temperatures of the edges $Y = -H$ and $Y = H$ are denoted by T_1 and T_2 respectively (Fig. 1). The problem of a uniform heat flow disturbed by an array of Griffith cracks in an infinite plate has been solved in [1]. In [1] the authors were able to find the stress intensity factors in terms of a simple quadrature. The problem presented in this paper does not have such an agreeable solution but reduces to that given in [1] once the appropriate specialization is made.

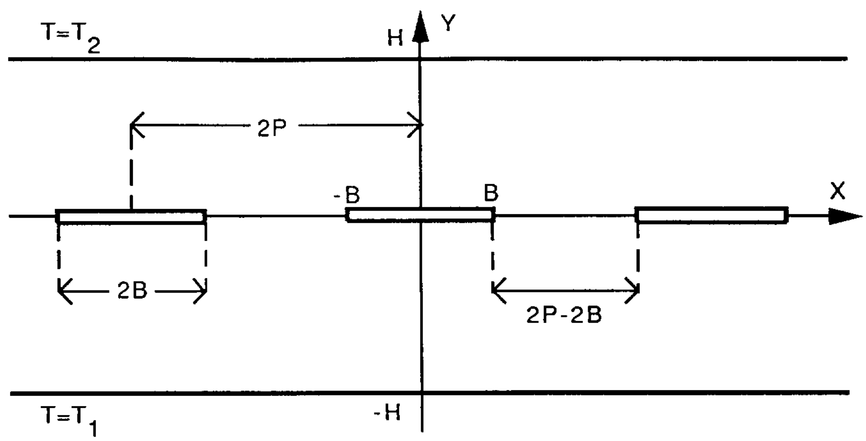

Fig. 1. Array of Griffith cracks parallel to the edges of a linear thermoelastic strip.

The notation used here is the same as that in [1]. Let T_0 denote the reference temperature κ the thermal conductivity, α the coefficient of linear expansion, ν Poisson's ratio, μ the shear modulus and E the Young's modulus of the material.

To non-dimensionalise the problem, the following quantities are introduced:

$$L = \frac{P}{\pi}\,,\ p_0 = \frac{\alpha E(T_1 - T_2)P}{4\pi H(1-\nu)}\,, \tag{1.1}$$

which have dimension of length and stress respectively.

The temperature at the point (X, Y) is denoted by

$$T(X, Y) = T_0\left[1 + \theta\left(\frac{X}{L}, \frac{Y}{L}\right)\right] \tag{1.2}$$

and as a matter of convenience we introduce the dimensionless variables

$$x = X/L,\ y = Y/L,\ b = B/L,\ h = H/L\,, \tag{1.3}$$

the dimensionless displacements

$$u(x, y) = \frac{2\mu U_X(X, Y)}{p_0 L}\,,\ v(x, y) = \frac{2\mu U_Y(X, Y)}{p_0 L}\,, \tag{1.4}$$

and the dimensionless stresses

$$\begin{aligned} s_{xx}(x, y) &= \frac{\sigma_{XX}(X, Y)}{p_0}\,,\ s_{yy} = \frac{\sigma_{YY}(X, Y)}{p_0}\,, \\ s_{xy}(x, y) &= \frac{\sigma_{XY}(X, Y)}{p_0}\,. \end{aligned} \tag{1.5}$$

The solution to our problem is achieved via Sneddon's general solution [2] which, in the notation used here, takes the form

$$\begin{aligned} u(x, y) &= \frac{\partial \chi}{\partial x} + \frac{\partial \phi}{\partial x} + (\beta^2 - 1)y\frac{\partial^2 \phi}{\partial x\,\partial y} + y\frac{\partial \psi}{\partial x}\,, \\ v(x, y) &= \frac{\partial \chi}{\partial y} - \beta^2\frac{\partial \phi}{\partial y} + (\beta^2 - 1)y\frac{\partial^2 \phi}{\partial y^2} + y\frac{\partial \psi}{\partial y} - \psi\,, \\ \theta(x, y) &= \delta^{-1}\frac{\partial \psi}{\partial y}\,, \\ s_{xx}(x, y) &= -\frac{\partial^2 \chi}{\partial y^2} - (\beta^2 - 1)\frac{\partial^2 \phi}{\partial y^2} - (\beta^2 - 1)y\frac{\partial^3 \phi}{\partial y^3} - y\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \psi}{\partial y}\,, \\ s_{yy}(x, y) &= \frac{\partial^2 \chi}{\partial y^2} - (\beta^2 - 1)\frac{\partial^2 \phi}{\partial y^2} + (\beta^2 - 1)y\frac{\partial^3 \phi}{\partial y^3} + y\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \psi}{\partial y}\,, \\ s_{xy}(x, y) &= \frac{\partial^2 \chi}{\partial x\,\partial y} + (\beta^2 - 1)y\frac{\partial^3 \phi}{\partial x\,\partial y^2} + y\frac{\partial^2 \psi}{\partial x\,\partial y}\,, \end{aligned} \tag{1.6}$$

where $\chi(x, y)$, $\phi(x, y)$ and $\psi(x, y)$ are arbitrary harmonic functions and

$$\beta^2 = \frac{2(1-\nu)}{1-2\nu}\,, \quad \delta = \frac{2\beta^2 h T_0}{T_1 - T_2}\,. \tag{1.7}$$

It is readily shown [3] that, in the absence of the cracks, the undisturbed thermoelastic field is given by

$$\begin{aligned}
&\theta^0(x, y) = \left[\frac{T_1 + T_2 + T_0}{2T_0}\right] - \left[\frac{T_1 - T_2}{2T_0}\right]\frac{y}{h}\,,\\
&v^0(x, y) = -y^2\,,\ s^0_{xx}(x, y) = 2y\,,\\
&u^0(x, y) = s^0_{xy}(x, y) = s^0_{yy}(x, y) = 0\,.
\end{aligned} \tag{1.8}$$

The presence of the cracks disturb (1.8) thereby yielding a new field

$$\begin{aligned}
&\theta(x, y) = \theta^0(x, y) + \theta^p(x, y)\,,\\
&v(x, y) = v^0(x, y) + v^p(x, y)\,,\\
&u(x, y) = u^p(x, y), s_{ij}(x, y) = s^p_{ij}(x, y)\,,
\end{aligned} \tag{1.9}$$

where the perturbations, denoted by the superscript p, can be found by solving the following mixed boundary value problem.

PROBLEM. Solve the dimensionless, plane strain equations of linear thermoelasticity in the region $0 < x < \pi$, $0 < y < h$ subject to the boundary conditions:

1. $u^p(0, y) = u^p(\pi, y) = s^p_{xy}(0, y) = s^p_{xy}(\pi, y) = 0\,;$ $\quad(0 < y < h)$
2. $s^p_{xy}(x, h) = s^p_{yy}(x, h) = 0\,;$ $\quad(0 < x < \pi)$
3. $s^p_{yy}(x, 0) = 0\,;$ $\quad(0 < x < \pi)$
4. $u^p(x, 0) = 0\,;$ $\quad(b < x < \pi)$

   $s^p_{xy}(x, 0) = 0\,;$ $\quad(0 < x < b)$
5. $\dfrac{\partial\theta^p}{\partial x}(0, y) = \dfrac{\partial\theta^p}{\partial x}(\pi, y) = 0\,;$ $\quad(0 < y < h)$
6. $\theta^p(x, h) = 0\,;$ $\quad(0 < x < \pi)$
7. $\theta^p(x, 0) = 0\,;$ $\quad(b < x < \pi)$

   $\dfrac{\partial\theta^p}{\partial y}(x, 0) = \dfrac{T_1 - T_2}{2hT_0}\,.$ $\quad(0 < x < b)$

The solution to this problem is obtained by superimposing the solutions to the following two mixed boundary value problems.

PROBLEM 1. Solve the dimensionless, plane strain equations of linear thermoelasticity in the region $0 < x < \pi$, $0 < y < h$ subject to the boundary conditions:

1. $u^1(0, y) = u^1(\pi, y) = s^1_{xy}(0, y) = s^1_{xy}(\pi, y) = 0$; $(0 < y < h)$
2. $s^1_{xy}(x, h) = s^1_{yy}(x, h) = 0$; $(0 < x < \pi)$
3. $u^1(x, 0) = s^1_{yy}(x, 0) = 0$; $(0 < x < \pi)$
4. $\dfrac{\partial \theta^1}{\partial x}(0, y) = \dfrac{\partial \theta^1}{\partial x}(\pi, y) = 0$; $(0 < y < h)$
5. $\theta^1(x, h) = 0$; $(0 < x < \pi)$
6. $\theta^1(x, 0) = 0$; $(b < x < \pi)$

   $\dfrac{\partial \theta^1}{\partial y}(x, 0) = \dfrac{T_1 - T_2}{2hT_0}$. $(0 < x < b)$

Calculate: $s^1_{xy}(x, 0) = f(x)$ $(0 < x < b)$

PROBLEM 2. Solve the dimensionless, plane strain equations of isothermal elasticity in the region $0 < x < \pi$, $0 < y < h$ subject to the boundary conditions:

1. $u^2(0, y) = u^2(\pi, y) = s^2_{xy}(0, y) = s^2_{xy}(\pi, y) = 0$; $(0 < y < h)$
2. $s^2_{xy}(x, h) = s^2_{yy}(x, h) = 0$; $(0 < x < \pi)$
3. $s^2_{yy}(x, 0) = 0$; $(0 < x < \pi)$
4. $u^2(x, 0) = 0$; $(b < x < \pi)$

   $s^2_{xy}(x, 0) = -f(x)$. $(0 < x < b)$

2. The solution of Problem 1

In Sneddon's general solution (1.6) the harmonic functions are chosen to be

$$\phi(x, y) = \frac{A_0 x^2}{2}\left(\frac{y}{h} - 1\right) + \frac{A_0 y^2}{2}\left(1 - \frac{y}{3h}\right) + \sum_{n=1}^{\infty} n^{-3} A_n \cosh(ny)\cos(nx)$$
$$+ \sum_{n=1}^{\infty} \frac{2n^{-3}A_n}{D(nh)}\left[\frac{\beta^2(nh\coth(nh) - 1)}{\beta^2 - 1} - \sinh^2(nh)\right]\sinh(ny)\cos(nx)\,; \tag{2.1}$$

$$\chi(x, y) = \frac{A_0(x^2 - y^2)}{2} - \sum_{n=1}^{\infty} n^{-3} A_n \cosh(ny)\cos(nx)$$
$$+ \sum_{n=1}^{\infty} \frac{2n^{-3}A_n}{D(nh)}[\sinh^2(nh) - (nh)^2]\sinh(ny)\cos(nx) \tag{2.2}$$

and

$$\psi(x, y) = -\beta^2\left[A_0 y + \frac{A_0(x^2 - y^2)}{2h} - \sum_{n=1}^{\infty} \frac{n^{-2}A_n \cosh(n(h - y))}{\sinh(nh)}\cos(nx)\right], \tag{2.3}$$

where

$$D(\xi) = \sinh(2\xi) - 2\xi . \tag{2.4}$$

All conditions of Problem 1 will be satisfied provided the weights $A_0, A_1, A_2, \ldots$ solve the following dual trigonometric equations

$$\begin{aligned} G_1(x) &= A_0 + \sum_{n=1}^{\infty} n^{-1} A_n \cos(nx) = 0 \qquad (b < x < \pi) , \\ F_1(x) &= \sum_{n=1}^{\infty} A_n \coth(nh) \cos(nx) = 1 \qquad (0 < x < b) . \end{aligned} \tag{2.5}$$

These dual equations are solved [4] by setting

$$A_0 = -\frac{1}{\pi} \int_0^b tp(t)\, dt , \tag{2.6}$$

$$A_n = -\frac{2}{\pi} \int_0^b p(t) \sin(nt)\, dt \qquad (n \geq 1) , \tag{2.7}$$

from which we find that

$$G_1(x) = -H(b - x) \int_x^b p(t)\, dt \qquad (0 < x < \pi) , \tag{2.8}$$

where $H(u)$ is the Heaviside step function. Furthermore, it can be shown that

$$F_1(x) = -\frac{1}{\pi} \int_0^b p(t) \left[\frac{t}{h} + \frac{2K}{\pi} Z\left(\frac{Kt}{\pi}\right) + \frac{\frac{2K}{\pi} \operatorname{sn} \frac{Kt}{\pi} \operatorname{cn} \frac{Kt}{\pi} \operatorname{dn} \frac{Kt}{\pi}}{\operatorname{sn}^2 \frac{Kt}{\pi} - \operatorname{sn}^2 \frac{Kx}{\pi}} \right] dt , \tag{2.9}$$

where $Z(\xi)$ is Jacobi's Zeta function and

$$K = \frac{\pi}{2} \left[1 + 2 \sum_{n=1}^{\infty} q^{n^2} \right]^2 , \quad q = e^{-h} \tag{2.10}$$

is the complete elliptic integral of the first kind while $\operatorname{sn}(\xi)$, $\operatorname{cn}(\xi)$, and $\operatorname{dn}(\xi)$ are Jacobian elliptic functions.

Hence $p(t)$ must solve the singular integral equation

$$\frac{1}{\pi} \int_0^b p(t) \frac{\frac{2K}{\pi} \operatorname{sn} \frac{Kt}{\pi} \operatorname{cn} \frac{Kt}{\pi} \operatorname{dn} \frac{Kt}{\pi}}{\operatorname{sn}^2 \frac{Kt}{\pi} - \operatorname{sn}^2 \frac{Kx}{\pi}}\, dt = B_1 - 1 \qquad (0 < x < b) \tag{2.11}$$

with subsidiary condition

$$p(0) = 0 , \tag{2.12}$$

where

$$B_1 - 1 = -\frac{1}{\pi}\int_0^b p(t)\left[\frac{t}{h} + \frac{2K}{\pi}\,Z\left(\frac{Kt}{\pi}\right)\right]\mathrm{d}t\,. \tag{2.13}$$

A suitable change of variables reduces (2.11) to the finite Hilbert transform [5], and hence yields

$$p(t) = \frac{(B_1 - 1)\,\mathrm{sn}\,\frac{Kt}{\pi}}{\left(\mathrm{sn}^2\,\frac{Kb}{\pi} - \mathrm{sn}^2\,\frac{Kt}{\pi}\right)^{1/2}}\,. \tag{2.14}$$

It follows at once that

$$\theta^1(x, 0) = \frac{T_1 - T_2}{2hT_0}\cdot\frac{\pi(B_1 - 1)F(\eta, \bar{k})}{K\,\mathrm{dn}\,\frac{Kb}{\pi}} \qquad (0 < x < b)\,, \tag{2.15}$$

where F is the incomplete elliptic integral of the first kind with parameters

$$\bar{k} = \frac{k'}{\mathrm{dn}\,\frac{Kb}{\pi}}\,,\ \sin(\eta) = \frac{\left(\mathrm{sn}^2\,\frac{Kb}{\pi} - \mathrm{sn}^2\,\frac{Kx}{\pi}\right)^{1/2}}{\mathrm{cn}\,\frac{Kx}{\pi}}\,. \tag{2.16}$$

Additionally, we obtain the flux

$$\frac{\partial\theta^1}{\partial y}(x, 0) = \frac{T_1 - T_2}{2hT_0}\left[1 + H(x - b)\,\frac{B_1 - 1}{\Delta_1(x)}\right], \tag{2.17}$$

where

$$\Delta_1(x) = \frac{\left(\mathrm{sn}^2\,\frac{Kx}{\pi} - \mathrm{sn}^2\,\frac{Kb}{\pi}\right)^{1/2}}{\mathrm{sn}\,\frac{Kx}{\pi}}\,. \tag{2.18}$$

Next, by virtue of (1.6) and (2.2) we observe that

$$f(x) = -\sum_{n=1}^{\infty} n^{-1}A_n[1 - M(nh)]\sin(nx)\,, \tag{2.19}$$

where

$$M(\xi) = \frac{1 - 2\xi + 2\xi^2 - \mathrm{e}^{-2\xi}}{D(\xi)}\,. \tag{2.20}$$

Therefore on making use of (2.7) and (2.8) it can be shown that

$$f(x) = -\frac{2hT_0}{T_1 - T_2} \cdot \frac{1}{\pi} \int_{-b}^{b} \theta^1(x, 0)\, K(t - x)\, \mathrm{d}t\,, \tag{2.21}$$

where

$$K(x) = \frac{1}{2}\cot\left(\frac{x}{2}\right) - \sum_{n=1}^{\infty} M(nh) \sin(nx)\,. \tag{2.22}$$

3. The solution of Problem 2

In Sneddon's general solution (1.6) we set

$$\phi(x, y) = (\beta^2 - 1)^{-1} \sum_{n=1}^{\infty} n^{-2} B_n \left[\cosh(ny) - \frac{2\sinh^2(nh)}{D(nh)} \sinh(ny)\right] \cos(nx)\,, \tag{3.1}$$

$$\chi(x, y) = \sum_{n=1}^{\infty} n^{-2} B_n \left[\cosh(ny) - \frac{2(\sinh^2(nh) - (nh)^2)}{D(nh)} \sinh(ny)\right] \cos(nx) \tag{3.2}$$

and

$$\psi(x, y) = 0\,. \tag{3.3}$$

Then all conditions of Problem 2 will be satisfied if the weights B_1, B_2, $B_3, \ldots$ solve the following dual trigonometric equations:

$$\begin{aligned} G_1(x) &= \sum_{n=1}^{\infty} n^{-1} B_n \sin(nx) = 0 && (b < x < \pi)\,, \\ F_2(x) &= \sum_{n=1}^{\infty} B_n [1 - M(nh)] \sin(nx) = -f(x) && (0 < x < b)\,. \end{aligned} \tag{3.4}$$

Let

$$B_n = \frac{2}{\pi} \int_0^b \frac{q(t)}{(b^2 - t^2)^{1/2}} \cos(nt)\, \mathrm{d}t \qquad (n \geq 1)\,, \tag{3.5}$$

then

$$G_2(x) = H(b - x) \int_0^x \frac{q(t)}{(b^2 - t^2)^{1/2}}\, \mathrm{d}t \qquad (0 < x < \pi)\,, \tag{3.6}$$

provided

$$\int_0^b \frac{q(t)}{(b^2 - t^2)^{1/2}}\, \mathrm{d}t = 0\,. \tag{3.7}$$

Additionally,

$$F_2(x) = -\frac{1}{\pi} \int_0^b \frac{q(t)}{(b^2 - t^2)^{1/2}} [K(t - x) - K(t + x)]\, \mathrm{d}t\,, \tag{3.8}$$

where $K(x)$ is given by (2.22). It follows at once that $q(t)$ must solve the singular integral equation

$$\frac{1}{\pi}\int_0^b \frac{q(t)}{(b^2-t^2)^{1/2}}\,[K(t-x)-K(t+x)]\,\mathrm{d}t = f(x) \qquad (0<x<b) \tag{3.9}$$

with subsidiary condition (3.7).

Using (2.21) and appealing to the symmetry of the problem, equations (3.9) and (3.7) can be re-written in the form

$$\frac{1}{\pi}\int_{-b}^b \frac{Q(t)}{(b^2-t^2)^{1/2}}\,K(t-x)\,\mathrm{d}t = 0 \qquad (-b<x<b)\,, \tag{3.10}$$

$$\int_{-b}^b \frac{Q(t)}{(b^2-t^2)^{1/2}}\,\mathrm{d}t = C_0\,, \tag{3.11}$$

where

$$Q(x) = q(x) + \frac{2hT_0}{T_1-T_2}\,\theta^1(x,0)(b^2-x^2)^{1/2} \tag{3.12}$$

and

$$C_0 = \frac{4hT_0}{T_1-T_2}\int_0^b \theta^1(x,0)\,\mathrm{d}x\,. \tag{3.13}$$

4. The stress intensity factor

The mode II stress intensity factor at the crack tip $(B,0)$ is defined by

$$k_{\mathrm{II}}(B) = -\lim_{X\to B^-}[2(B-X)]^{1/2}\,\frac{E}{2(1-\nu^2)}\,\frac{\partial U_X^2}{\partial X}(X,0)\,, \tag{4.1}$$

from which it is not difficult to show that

$$k_{\mathrm{II}}(B) = p_0\left[\frac{L}{b}\right]^{1/2} Q(b)\,. \tag{4.2}$$

It is known [6] that when a crack of length $2B$ in an infinite thermoelastic solid disturbs a uniform heat flow of strength $(T_1-T_2)\kappa/2H$ the stress intensity factor is given by

$$k_0 = -\frac{1}{2}\,p_0 L^{1/2} b^{3/2}\,. \tag{4.3}$$

Using this infinite solution we obtain the scaled stress intensity factor

$$\frac{k_{\mathrm{II}}(B)}{k_0} = -\frac{2Q(b)}{b^2}\,. \tag{4.4}$$

5. Numerical solution of the integral equation

Equations (3.10) and (3.11) are most easily solved by the method of Ioakimidis and Theocaris [7]. In this method (3.10) and (3.11) are replaced by the linear algebraic system

$$\frac{1}{\pi}\sum_{k=1}^{N} H_k Q(t_k) K(t_k - x_j) = 0 \quad j = 1, 2, \ldots, N-1 ,$$
$$\sum_{k=1}^{N} H_k Q(t_k) = C_0 , \tag{5.1}$$

with weights

$$H_k = \frac{\pi}{N-1}\begin{cases} 1/2 , & k = 1, N \\ 1 , & k = 2, 3, \ldots, N-1 . \end{cases} \tag{5.2}$$

Also,

$$t_k = b\cos\left(\frac{k-1}{N-1}\right)\pi \quad k = 1, 2, \ldots, N \tag{5.3}$$

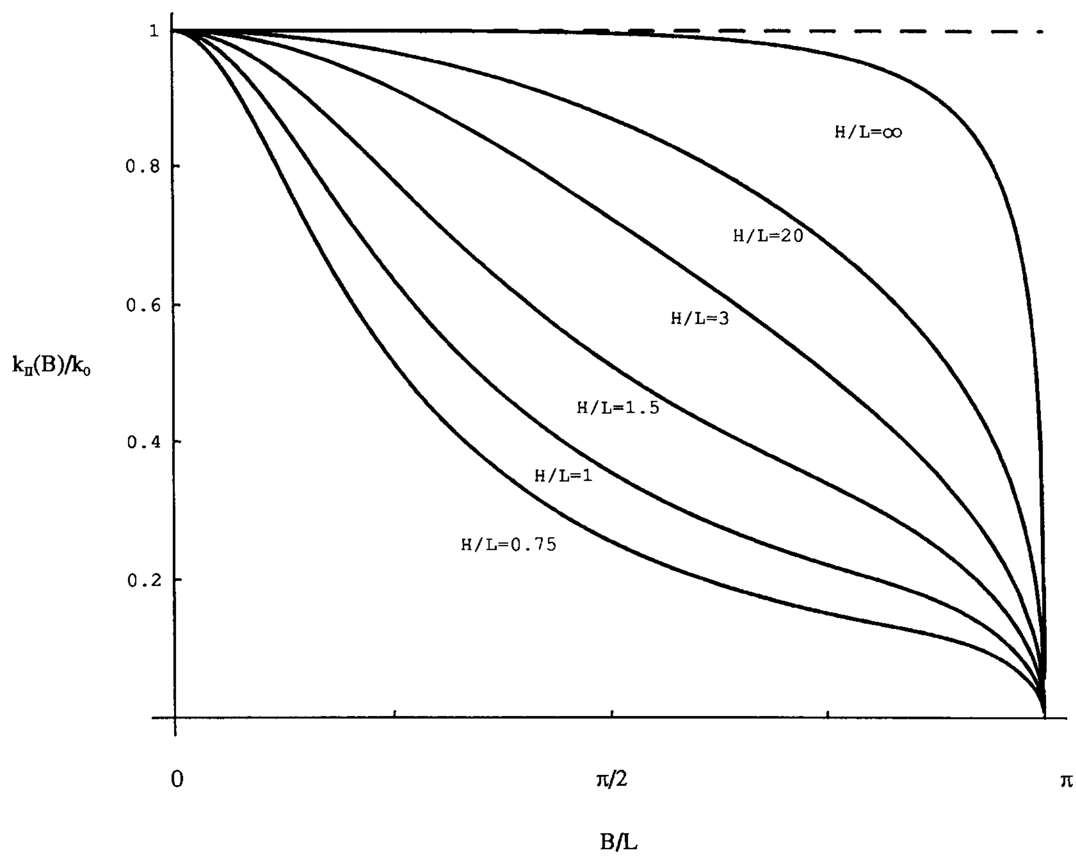

Fig. 2. The variation of $k_{II}(B)/k_0$ with B/L and H/L.

and

$$x_j = b\cos\left(\frac{j-\frac{1}{2}}{N-1}\right)\pi \quad j = 1, 2, \ldots, N-1 \tag{5.4}$$

are the zeros of the Chebyshev polynomials $T_{N-1}(x)$ and $U_{N-2}(x)$ respectively.

Once the system (5.1) has been solved, the stress intensity factor is determined by (4.4) and the results of such a computation are recorded in Fig. 2 which illustrates how the stress intensity varies with B/L and H/L. In the limiting cases as $B/L \to 0$ and $B/L \to \pi$ we recover the expected results that $k_{II}(B)/k_0$ approaches one and zero respectively. It is observed that as the strip width is reduced the sliding of the crack surfaces over one another becomes more pronounced. Results for the problem of an infinite sheet containing an array of cracks [1] are included for comparison. The solution presented in this paper reduces to that given in [1] as the specialization $H/L \to \infty$ is made.

References

1. G. Melrose and S. Davidson, Uniform heat flow past a collinear array of Griffith cracks. In G.A.C. Graham and S.K. Milik (eds), *Continuum Mechanics and its Applications*. New York: Hemisphere Publishing Corp. (1989).
2. I.N. Sneddon, Solutions of the equations of thermoelastic equilibrium. *Arch. Mech. Stos.* 14 (1962) 113–125.
3. H. Parkus, *Thermoelasticity*. Massachusetts: Blaisdell (1968).
4. G. Melrose, Doctoral Dissertation. Old Dominion University, Norfolk, Virginia, USA (1984).
5. F.G. Tricomi, On the finite Hilbert transformation. *Quart. J. Math.* 2 (1951) 199–211.
6. A.L. Florence and J.N. Goodier, Thermal stresses due to the disturbance of uniform heat flow by an insulated ovaloid hole. J. Appl. Mech. 27 (1960) 635–639.
7. P.S. Theocaris and N.I. Ioakimidis, Numerical integration methods for the solution of singular integral equations. *Quart. J. Math.* 35 (1977) 173–183.